

PARSEVAL FRAMES FOR ICC GROUPS

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ABSTRACT. We analyze Parseval frames generated by the action of an ICC group on a Hilbert space. We parametrize the set of all such Parseval frames by operators in the commutant of the corresponding representation. We characterize when two such frames are strongly disjoint. We prove an undersampling result showing that if the representation has a Parseval frame of equal norm vectors of norm $\frac{1}{\sqrt{N}}$, the Hilbert space is spanned by an orthonormal basis generated by a subgroup. As applications we obtain some sufficient conditions under which a unitary representation admits a Parseval frame which is spanned by an Riesz sequences generated by a subgroup. In particular, every subrepresentation of the left regular representation of a free group has this property.

CONTENTS

1. Introduction	1
2. General theory: A new parametrization theorem	5
3. Parseval frames and subgroups	11
References	17

1. INTRODUCTION

Frames play a fundamental role in signal processing, image and data compression and sampling theory. They provide an alternative to orthonormal bases, and have the advantage of possessing a certain degree of redundancy which can be useful in applications, for example when data is lost during transmission. Also, frames can be better localized, a feature which lead to the success of Gabor frames and wavelet theory (see e.g., [Dau92]).

The term “frame” was introduced by Duffin and Schaffer [DS52] in their study of non-harmonic Fourier series, and has generated important research areas and remarkable breakthroughs [OCS02]. Recent results show that frames can provide a universal language in which many fundamental problems in pure mathematics can be formulated: the Kadison-Singer problem in operator algebras, the Bourgain-Tzafriri conjecture in Banach space theory, paving Töplitz operators in harmonic analysis and many others (see [CFTW06] for an excellent account).

Definition 1.1. Let \mathcal{H} be a Hilbert space. A family of vectors $\{e_i \mid i \in I\}$ in \mathcal{H} is called a *frame* if there exist constants $A, B > 0$ such that for all $f \in \mathcal{H}$,

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, e_i \rangle|^2 \leq B\|f\|^2.$$

Research supported in part by a grant from the National Science Foundation DMS-0704191
2000 *Mathematics Subject Classification.* 46L10, 42C20.

Key words and phrases. Parseval frames, Kadison-Singer problem, undersampling, II_1 -factors.

If $A = B = 1$, then $\{e_i | i \in I\}$ is called a *Parseval frame*. If we only require the right hand inequality, then we say that $\{e_i | i \in I\}$ is a *Bessel sequence*.

In their seminal paper by Han and Larson [HL00], operator theoretic foundations for frame theory and group representations were formulated. One of the key observations in their paper was that every Parseval frame is the orthogonal projection of an orthonormal basis. This lead them to the notion of *disjointness* of frames.

Definition 1.2. Let $\{e_i | i \in I\}, \{f_i | i \in I\}$ be two Parseval frames for the Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 respectively. The Parseval frames are called *strongly disjoint* if $\{e_i \oplus f_i | i \in I\}$ is a Parseval frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$. Similarly, if we have $\{e_i^1 | i \in I\}, \dots, \{e_i^N | i \in I\}$ Parseval frames for the Hilbert spaces \mathcal{H}_i respectively, we say that these frames form an *N-tuple of strongly disjoint Parseval frames* if $\{e_i^1 \oplus \dots \oplus e_i^N | i \in I\}$ is a Parseval frame for $\mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_N$.

The Parseval frames are called *unitarily equivalent* if there exists a unitary operator $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that $Ue_i = f_i$ for all $i \in I$.

If the direct sum of two Parseval frames is an orthonormal basis, we say that one is the *complement* of the other, or one *complements* the other.

Many properties of frames are encoded in the associated *frame transform* (or *analysis operator*). It is the operator that associates to a vector its coefficients in the given frame.

Definition 1.3. Let $\mathcal{E} := \{e_i | i \in I\}$ be a Bessel sequence in a Hilbert space \mathcal{H} . The operator $\Theta_{\mathcal{E}} : \mathcal{H} \rightarrow l^2(I)$ defined by

$$\Theta_{\mathcal{E}}(f) = (\langle f, e_i \rangle)_{i \in I}, \quad (f \in \mathcal{H})$$

is called the *analysis operator* or *frame transform* associated to \mathcal{E} .

Strong disjointness can be characterized in terms of frame transforms as follows:

Proposition 1.4. [HL00] *Let $\mathcal{E} := \{e_i | i \in I\}$ and $\mathcal{F} := \{f_i | i \in I\}$ be two Parseval frames for the Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 respectively. Then \mathcal{E} and \mathcal{F} are unitarily equivalent iff the frame transforms $\Theta_{\mathcal{E}}$ and $\Theta_{\mathcal{F}}$ have the same range. The following affirmations are equivalent:*

- (i) *The Parseval frames \mathcal{E} and \mathcal{F} are strongly disjoint.*
- (ii) *The frame transforms $\Theta_{\mathcal{E}}$ and $\Theta_{\mathcal{F}}$ have orthogonal ranges.*
- (iii) $\Theta_{\mathcal{F}}^* \Theta_{\mathcal{E}} = 0$.
- (iv) *For all $v_1 \in \mathcal{D}_1, v_2 \in \mathcal{D}_2$, where $\mathcal{D}_1, \mathcal{D}_2$ are dense in $\mathcal{H}_1, \mathcal{H}_2$ respectively,*

$$\sum_{i \in I} \langle v_1, e_i \rangle \overline{\langle v_2, f_i \rangle} = 0.$$

In what follows we will call two Bessel sequences (with same index set) *strongly disjoint* if the range spaces of their analysis operators are orthogonal.

In this paper we will be interested only in the Parseval frames generated by the action of an infinite-conjugacy-classes (ICC) group on a Hilbert space. The ICC property implies that the associated left-regular representation is a II_1 factor, and we can use the theory of factor von Neumann algebras [JS97]. Our purpose is to follow and further the results in [HL00].

After the Han-Larson paper, frames for abstract Abelian groups have been studied in [TW06, ALTW04] using Pontryagin duality. As we shall see, Parseval frames for ICC groups have quite different properties. Many strongly disjoint Parseval frames can be found in the same representation (Theorem 2.2), and there are undersampling results in some cases (Proposition 3.1 and Theorem 3.3). Frames for ICC groups fit into the more general theory

of group-like unitary systems, a theory which has the Gabor (or Weyl-Heisenberg) frames as the central example, see [GH03, HL01, Han04] for details.

Definition 1.5. Let G be a countable group. Let π be a unitary representation of G on the Hilbert space \mathcal{H} . A vector $\xi \in \mathcal{H}$ is called a *frame/Parseval frame/ONB vector* for \mathcal{H} (with the representation π) iff $\{\pi(g)\xi \mid g \in G\}$ is a frame/Parseval frame/ONB for \mathcal{H} .

Two Parseval frame vectors ξ, η for \mathcal{H} are called *unitarily equivalent/strongly disjoint* if the corresponding Parseval frames $\{\pi(g)\xi \mid g \in G\}$ and $\{\pi(g)\eta \mid g \in G\}$ are unitarily equivalent/strongly disjoint. Similarly for an N -tuple of strongly disjoint Parseval frame vectors.

As shown in [HL00], every representation of a group that has a Parseval frame vector is isomorphic to a sub-representation of the left-regular representation (Proposition 1.8).

Definition 1.6. Let G be a countable group. The *left-regular representation* λ of G is defined on $l^2(G)$ by

$$(\lambda(g)\xi)(h) = \xi(g^{-1}h), \quad (\xi \in l^2(G), h \in G, g \in G).$$

Equivalently, if $\delta_g, g \in G$ is the canonical orthonormal basis for $l^2(G)$, then

$$\lambda(g)\delta_h = \delta_{gh}, \quad (h, g \in G).$$

The von Neumann algebra generated by the unitary operators $\lambda(g), g \in G$ is denoted by $L(G)$.

The *right-regular representation* ρ of G is defined on $l^2(G)$ by

$$(\rho(g)\xi)(h) = \xi(hg), \quad (\xi \in l^2(G), h \in G, g \in G).$$

Equivalently,

$$\rho(g)\delta_h = \delta_{hg^{-1}}, \quad (h, g \in G).$$

The group G is called an ICC (infinite conjugacy classes) group if, for all $g \in G, g \neq e$, the set $\{hgh^{-1} \mid h \in G\}$ is infinite.

The commutant of the von Neumann algebra $L(G)$ is the von Neumann algebra generated by the right-regular representation ρ . When the group is ICC, $L(G)$ is a II_1 factor (see [JS97]).

The paper is organized as follows: in Section 2, we study the general properties of Parseval frames for ICC groups. In Theorem 2.2 we show how a Parseval frame vector can be complemented by several other Parseval frame vectors for the same representation and a “remainder” Parseval frame vector for a subspace. While Lemma 2.3 characterizes strongly disjoint Parseval frame vectors in terms of their cyclic projections we present some new properties along the same lines in Proposition 2.5.

We give a parametrization of Parseval frame vectors in Theorem 2.6. Another such parametrization was given in [HL00], see Theorem 1.9. The advantage of our parametrization is that it uses operators in the commutant in the representation, so it can be extended to *all* the vectors of the frame. Theorem 2.7 characterizes strong disjointness and unitary equivalence in terms of this parametrization.

In Section 3 we will be interested in the relation between Parseval frames and subgroups. We will work with Parseval frame vectors that have norm-square equal to $\frac{1}{N}$ with $N \in \mathbb{N}$ and we will assume that there exists a subgroup H of index N . We give two “undersampling” results: in Proposition 3.1, we show that in this situation, there exist orthonormal bases generated by the action of the subgroup H , and in Theorem 3.3 we show that, if in addition

H is normal, then we can construct N strongly disjoint Parseval frame vectors such that, when undersampled to the subgroup, they form orthonormal bases (up to a multiplicative constant). We apply these results to the Feichtinger frame decomposition problem for free groups to get that every frame representation for free groups admits a frame that is a finite union of Riesz sequences.

Definition 1.7. We will use the following notations: if \mathcal{A} is a set of operators on a Hilbert space \mathcal{H} , then \mathcal{A}' is the commutant of \mathcal{A} , i.e., the set of all operators that commute with all the operators in \mathcal{A} . \mathcal{A}'' is the double-commutant, i.e., the commutant of \mathcal{A}' . By von Neumann's double commutant theorem, \mathcal{A}'' coincides with the von Neumann algebra generated by \mathcal{A} . Two (orthogonal) projections p and q in a von Neumann algebra \mathcal{A} are said to be equivalent (denoted by $p \sim q$) if there exists an operator (partial isometry) $u \in \mathcal{A}$ such that $uu^* = p$ and $u^*u = q$. A von Neumann algebra \mathcal{A} is finite if there is no proper projection of I that is equivalent to I in \mathcal{A} . We refer to [KR97] for more details and some properties about von Neumann algebras that will be used in the rest of the paper.

If π is a representation of a group G on a Hilbert space \mathcal{H} , and $N \in \mathbb{N}$, then π^N is the representation of G on \mathcal{H}^N defined by

$$\pi^N(g) = \underbrace{\pi(g) \oplus \cdots \oplus \pi(g)}_{N \text{ times}}, \quad (g \in G).$$

If p' is a projection in $\pi(G)'$, then $\pi p'$ is the representation of G on $p'\mathcal{H}$ defined by

$$(\pi p')(g) = p'\pi(g)p', \quad (g \in G).$$

If M is a von Neumann algebra with a trace, we will denote the trace by tr_M

The next proposition proved by Han and Larson in [HL00] is the starting point for the theory of Parseval frames for groups. It shows that any Parseval frame generated by the representation of a group is in fact isomorphic to the projection of the canonical basis in the left-regular representation of the group. The isomorphism is in fact the frame transform, the projection is its range and it lies in the commutant of the left-regular representation.

Proposition 1.8. [HL00] *Let G be a countable ICC group and let $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ be a unitary representation of the group G on the Hilbert space \mathcal{H} . Suppose $\xi \in \mathcal{H}$ is a Parseval frame vector for \mathcal{H} . Then:*

- (i) *The frame transform Θ_ξ is an isometric isomorphism between \mathcal{H} and the subspace $p'l^2(G)$, where $p' := \Theta_\xi \Theta_\xi^*$.*
- (ii) *The frame transform Θ_ξ intertwines the representations π on \mathcal{H} and λ on $l^2(G)$, i.e., $\Theta_\xi \pi(g) = \lambda(g) \Theta_\xi$, for all $g \in G$. The projection $p' := \Theta_\xi \Theta_\xi^*$ commutes with $\lambda(G)$.*
- (iii) *$\Theta_\xi \xi = p' \delta_e$, $\Theta_\xi^* \delta_e = \xi$. The trace of p' in $L(G)'$ is $\text{tr}_{L(G)'}(p') = \|\xi\|^2$.*

Thus the Parseval frame $\{\pi(g)\xi \mid g \in G\}$ for \mathcal{H} is unitarily equivalent to the Parseval frame $\{\lambda(g)p'\delta_e \mid g \in G\}$ for $p'l^2(G)$, via the frame transform Θ_ξ .

In [HL00], the authors proved that Parseval vectors can be parametrized by unitary operators in the algebra $\pi(G)''$. Note that these operators are *not* in the commutant, and this has the drawback that one can not map the entire Parseval frame into the other. We will give an alternative parametrization, that uses operators in the commutant in Proposition 2.6.

Theorem 1.9. [HL00] *Let G be a countable ICC group and let $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ be a unitary representation of the group G on the Hilbert space \mathcal{H} . Suppose $\xi \in \mathcal{H}$ is a Parseval frame vector for \mathcal{H} . Then $\eta \in \mathcal{H}$ is a Parseval frame vector for \mathcal{H} if and only if there exists a unitary $u \in \pi(G)''$ such that $u\xi = \eta$. In particular $\|\eta\| = \|\xi\|$.*

2. GENERAL THEORY: A NEW PARAMETRIZATION THEOREM

We will consider an ICC group G and $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ a unitary representation of the group on the Hilbert space \mathcal{H} . We will assume that this representation has a Parseval frame vector ξ_1 . The main goal of this section is to obtain an alternate parametrization of all the Parseval frame vectors by using operator vectors with entries in the commutant of $\pi(G)$. Although the focus of this paper is on ICC groups, the new parametrization result works for other groups for which we will discuss the details at the end of this section.

The norm-square of the vector $\|\xi_1\|^2$ fits into one N times, and there may be some remainder $0 \leq r \leq \|\xi_1\|^2$. The next theorem shows that we can complement the Parseval frame vector ξ_1 by $N - 1$ Parseval frame vectors for \mathcal{H} and one “remainder” Parseval frame vector for a subspace of \mathcal{H} . Moreover, the complementing procedure works also if we start with several strongly disjoint Parseval frame vectors for \mathcal{H} . Of course, if $\|\xi_1\|^2 = \frac{1}{N}$ with $N \in \mathbb{N}$, the remainder Parseval vector is not needed and can be discarded. Our results have a simpler statement if the extra assumption $\|\xi_1\|^2 = \frac{1}{N}$ is added, and the remainders disappear. We recommend the reader do this for an easier understanding of the statements. However we sacrificed (part of) the aesthetics for generality.

Lemma 2.1. [HL00, GH04] *Let G be a countable ICC group and let $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ be a unitary representation of the group G on the Hilbert space \mathcal{H} . Suppose $\xi \in \mathcal{H}$ is a Parseval frame vector for \mathcal{H} and let $p'_\xi = \Theta_\xi \Theta_\xi^*$ and p' be a projection in $L(G)'$. Then*

- (i) $p' \sim p'_\xi$ in the von Neumann algebra $L(G)'$ if and only if there exists a Parseval frame vector η for \mathcal{H} such that $p' = p'_\eta$;
- (ii) p' is equivalent to a subprojection of p'_ξ if and only if there exists a Parseval frame vector η for $\overline{\text{span}}\{\pi(g)\eta : g \in G\}$ such that $p' = p'_\eta$;

Theorem 2.2. *Let G be a countable ICC group and let $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ be a unitary representation of the group G on the Hilbert space \mathcal{H} . Suppose $\xi_1 \in \mathcal{H}$ is a Parseval frame vector for \mathcal{H} . Let $N := \left\lfloor \frac{1}{\|\xi_1\|^2} \right\rfloor \in \mathbb{N}$.*

- (i) *There exist $\xi_2, \dots, \xi_N, \xi_{N+1} \in \mathcal{H}$ with the following properties:*
 - (a) ξ_2, \dots, ξ_N are Parseval frame vectors for \mathcal{H} .
 - (b) $\|\xi_{N+1}\|^2 = 1 - N\|\xi_1\|^2 =: r < \|\xi_1\|^2$, and there exists a projection $p'_r \in \pi(G)'$ such that $\{\pi(g)\xi_{N+1} \mid g \in G\}$ is a Parseval frame for $p'_r\mathcal{H}$, so ξ_{N+1} is a Parseval frame vector for $p'_r\mathcal{H}$ with the representation $\pi^r := \pi p'_r$.
 - (c) ξ_1, \dots, ξ_{N+1} is a strongly disjoint $N + 1$ -tuple of Parseval frame vectors.
- (ii) *If ξ_1, \dots, ξ_{N+1} are as in (i) then there is no vector ξ_{N+2} such that ξ_{N+2} is a Parseval frame vector for some representation π_{N+2} of G , and ξ_{N+2} is strongly disjoint from all ξ_i , $i = 1, N + 1$.*
- (iii) *If $1 \leq M \leq N$ and ξ_1, \dots, ξ_M is a strongly disjoint M -tuple of Parseval frame vectors for \mathcal{H} , then there exist $\xi_{M+1}, \dots, \xi_N, \xi_{N+1}$ such that the properties in (i) are satisfied.*

Proof. (i) Let $p'_1 = \Theta_{\xi_1} \Theta_{\xi_1}^*$. Since $L(G)'$ is a factor von Neumann algebra, we have that there exist mutually orthogonal projections $p'_2, \dots, p'_N, p'_{N+1}$ in $L(G)'$ such that $p'_i \sim p'_1$ for $i = 2, \dots, N$, p'_{N+1} is equivalent to a subprojection of p'_1 and $\sum_{i=1}^{N+1} p'_i = I$. By Lemma 2.1, there exist $\xi_2, \dots, \xi_N, \xi_{N+1} \in \mathcal{H}$ such that ξ_2, \dots, ξ_N are Parseval frame vectors for \mathcal{H} with $p'_i = p'_{\xi_i}$, and ξ_{N+1} has the property that $\{\pi(g)\xi_{N+1} \mid g \in G\}$ is a Parseval frame for its closed linear span. Since $p'_1, \dots, p'_N, p'_{N+1}$ are mutually orthogonal, we get that ξ_1, \dots, ξ_{N+1} is a strongly disjoint $N+1$ -tuple of Parseval frame vectors. Moreover, from $\sum_{i=1}^{N+1} p'_i = I$ we obtain that $\xi_1 \oplus \dots \oplus \xi_N \oplus \xi_{N+1}$ has norm one and hence it is an ONB vector for $\pi^N \oplus \pi^r$, where $\pi^r = \pi p'_r$ and p'_r is the orthogonal projection onto $\overline{\text{span}}\{\pi(g)\xi_{N+1} \mid g \in G\}$.

(ii) If ξ_{N+2} is strongly disjoint from all ξ_i , $i = 1, N$, then $\zeta := (\xi_1 \oplus \dots \oplus \xi_{N+1}) \oplus \xi_{N+2}$ is a Parseval frame vector with $\|\zeta\| > 1$, and this is impossible.

(iii) This clearly is a more general form of (i), and the proof is exactly as in the proof of (i). \square

The following two more lemmas are needed in the proof of Theorem 2.6.

Lemma 2.3. [HL08] *Let G be a countable group and let $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ be a unitary representation of the group G on the Hilbert space \mathcal{H} . Assume there exists a Parseval frame vector ξ_1 for \mathcal{H} . Suppose η_1, η_2 are two Parseval frame vectors for some subspaces of \mathcal{H} . Let p_{η_i} be the projection onto the subspace $\pi(G)'\eta_i$, $i = 1, 2$. Then*

(i) *the two Parseval frame vectors η_1, η_2 are strongly disjoint if and only if projections p_{η_1}, p_{η_2} are orthogonal.*

(ii) *the two Parseval frame vectors η_1, η_2 are unitarily equivalent if and only if $p_{\eta_1} = p_{\eta_2}$.*

Lemma 2.4. [GH01] *Let G be a countable group and let $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ be a unitary representation of the group G on the Hilbert space \mathcal{H} . Assume there exists a Parseval frame vector ξ_1 for \mathcal{H} . Suppose η is a Parseval frame vector for a subspace of \mathcal{H} . Then there exists a vector ζ such that $\eta + \zeta$ is a Parseval frame vector for \mathcal{H} , and η and ζ are strongly disjoint Parseval frame vectors.*

Proposition 2.5. *Let G be a countable ICC group and let $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ be a unitary representation of the group G on the Hilbert space \mathcal{H} . Assume that there exists a Parseval frame vector ξ_1 for \mathcal{H} . For $\xi \in \mathcal{H}$, let p_ξ be the projection onto the subspace $\pi(G)'\xi$.*

(i) *If η is a Parseval frame vector for a subspace for \mathcal{H} then $\text{tr}_{\pi(G)''}(p_\eta) = \|\eta\|^2$.*

(ii) *If η is a Parseval frame vector for \mathcal{H} and u is a unitary operator in $\pi(G)''$, then the Parseval frame vector $u\eta$ is strongly disjoint from η iff $u\eta$ is orthogonal to the range of p_η , and in this case $p_\eta \perp p_{u\eta}$.*

(iii) *Let $N \geq 1, N \in \mathbb{Z}$, and suppose ξ_1, \dots, ξ_{N+1} is a strongly disjoint $N+1$ -tuple of Parseval frame vectors for some subspaces of \mathcal{H} , with $\sum_{i=1}^{N+1} \|\xi_i\|^2 = 1$. Then the projections $p_{\xi_i}, i = 1, \dots, N$ are mutually orthogonal, and $p_{\xi_1} + \dots + p_{\xi_{N+1}} = 1$.*

Proof. By Proposition 1.8, we can assume $\mathcal{H} = p'l^2(G)$ and $\pi = \lambda p'$ for some projection $p' \in L(G)'$ with $\text{tr}_{L(G)'}(p') = \|\xi_1\|^2$.

Let p'_η be the projection onto the subspace $\overline{\pi(G)''\eta}$ generated by the Parseval frame vector η . Then $p_\eta \in \pi(G)'$ and by [JS97, Remark 2.2.5] we have

$$\dim_{\pi(G)''} \mathcal{H} = \frac{\text{tr}_{\pi(G)''}(p_\eta)}{\text{tr}_{\pi(G)'}(p'_\eta)}.$$

But, by [JS97, Proposition 2.2.6(vi)],

$$\dim_{\pi(G)''} \mathcal{H} = \dim_{L(G)p'}(p'l^2(G)) = \operatorname{tr}_{L(G)'}(p') \dim_{L(G)} l^2(G) = \|\xi_1\|^2.$$

On the other hand, using the uniqueness of the trace on factors, we have

$$\operatorname{tr}_{\pi(G)'}(p'_\eta) = \operatorname{tr}_{p'L(G)'p'}(p'_\eta) = \frac{\operatorname{tr}_{L(G)'}(p'_\eta)}{\operatorname{tr}_{L(G)'}(p')} = \frac{\|\eta\|^2}{\|\xi_1\|^2}.$$

We used the fact that $p'_\eta \in L(G)'$ is the projection onto the subspace in $\mathcal{H} \subset l^2(G)$ spanned by the Parseval frame $\{\pi(g)\eta = \lambda(g)\eta \mid g \in G\}$, and therefore, by Proposition 1.8, $\operatorname{tr}_{L(G)'}(p'_\eta) = \|\eta\|^2$. From these equalities, (i) follows.

To prove (ii), we use Theorem 1.9 to see that $u\eta$ is a Parseval frame vector. By Lemma 2.3, the Parseval frame vectors are disjoint iff p_η and $p_{u\eta}$ are orthogonal. Thus one implication is trivial. If $u\eta$ is orthogonal to the range of p_η , then for all $x', y' \in \pi(G)'$ we have $\langle x'u\eta, y'\eta \rangle = \langle u\eta, x'^*y'\eta \rangle = 0$ so $p_{u\eta}$ is perpendicular to p_η .

For (iii), with Lemma 2.3, we have that p_{ξ_i} are mutually orthogonal. From (i), we have that

$$\operatorname{tr}_{\pi(G)''}(\sum_{i=1}^{N+1} p_i) = \sum_{i=1}^{N+1} \|\xi_i\|^2 = 1,$$

therefore $\sum_{i=1}^{N+1} p_i = 1$. □

Now we are ready to parametrize the Parseval frame vectors for \mathcal{H} . As we mentioned before, such a result was given in [HL00], see Theorem 1.9. The drawback of their result is that they are using operators in the von Neumann algebra $\pi(G)''$ itself, not in its commutant. We give here a parametrization that uses operators in the commutant. As Han and Larson proved in their paper, we cannot expect to use unitary operators in $\pi(G)'$. Instead, we will use the $N + 1$ strongly disjoint Parseval frame vectors given by Theorem 2.2 and $N + 1$ operators in the commutant $\pi(G)'$ that satisfy the orthogonality relation (2.2) which in fact represents the norm-one property of a first row in a unitary matrix of operators.

Theorem 2.6. *Let G be a countable ICC group and let $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ be a unitary representation of the group G on the Hilbert space \mathcal{H} . Suppose $\xi_i, i = 1, \dots, N + 1$ is a strongly disjoint $N + 1$ -tuple of Parseval frame vectors as in Theorem 2.2(i). Let $\eta \in \mathcal{H}$ be another Parseval frame vector for a subspace of \mathcal{H} , and let q' be the projection onto this subspace. Then there exist unique $u'_i \in \pi(G)'$, $i = 1, \dots, N + 1$, with $q'u'_i = u'_i$, $i = 1, \dots, N + 1$, and $u'_{N+1}p'_r = u'_{N+1}$ (where p'_r is the projection onto the span of the Parseval frame generated by the vector ξ_{N+1} , as in Theorem 2.2) such that*

$$(2.1) \quad \eta = u'_1\xi_1 + \dots + u'_{N+1}\xi_{N+1}.$$

Moreover

$$(2.2) \quad \sum_{i=1}^{N+1} u'_i u'^{*}_i = q'.$$

Conversely, if the vector η is defined by (2.1) with $u'_i \in L(G)'$, $q'u'_i = u'_i$, $i = 1, \dots, N + 1$, and $u'_{N+1}p'_r = u'_{N+1}$ satisfying (2.2), then η is a Parseval frame vector for \mathcal{H} .

Proof. We prove the theorem first for the case when $q' = 1$, so η is a Parseval frame vector for the entire space \mathcal{H} .

By Theorem 2.2(iii), there exist $\eta_2, \dots, \eta_{N+1} \in \mathcal{H}$ that together with $\eta_1 := \eta$ form a strongly disjoint $N+1$ -tuple of Parseval frame vectors as in Theorem 2.2(i). Then $\eta_1 \oplus \dots \oplus \eta_{N+1}$ is an ONB vector for $\pi^N \oplus \pi p'_r = \lambda$. Then there exists a unitary $u' \in L(G)'$ such that $u'(\xi_1 \oplus \dots \oplus \xi_{N+1}) = \eta_1 \oplus \dots \oplus \eta_{N+1}$.

Let p'_i be the projections onto the i -th component. We can identify $p'_{N+1} = p'_r$, and in our case $q' = p'_1$. Let $u'_i := p'_1 u' p'_i$. Then $u'_i \in \pi(G)'$, $u'_{N+1} p'_r = u'_{N+1}$, and

$$\sum_{i=1}^{N+1} u'_i \xi_i = \sum_{i=1}^{N+1} p'_1 u' p'_i (\xi_1 \oplus \dots \oplus \xi_{N+1}) = p'_1 u' \left(\sum_{i=1}^{N+1} p'_i \right) (\xi_1 \oplus \dots \oplus \xi_{N+1}) = \eta_1.$$

This proves (2.1).

To prove uniqueness, suppose $\sum_{i=1}^{N+1} v'_i \xi_i = 0$ for some operators $v'_i \in \pi(G)'$, with $v'_{N+1} p'_r = v'_{N+1}$. Then, for all $g \in G$, $\pi(g) \sum_{i=1}^{N+1} v'_i \xi_i = 0$. By Lemma 2.3, the vectors $v'_i \xi_i$ are mutually orthogonal. Since $\pi(g)$ is unitary, it follows that $\pi(g) v'_i \xi_i = 0$ for all $i = 1, \dots, N+1$. Therefore $v'_i \pi(g) \xi_i = 0$ for all $g \in G$, $i = 1, N+1$. But $\pi(g) \xi_i$ span \mathcal{H} for $i = 1, \dots, N$, and span $p'_r \mathcal{H}$ for $i = N+1$. Therefore $v'_i = 0$ for all $i = 1, \dots, N+1$. This implies the uniqueness.

We check now (2.2). We have

$$q' = p'_1 = p'_1 u' u'^* p'_1 = p'_1 u' \left(\sum_{i=1}^{N+1} p_i \right) u'^* p'_1 = \sum_{i=1}^{N+1} p'_1 u' p'_i (p'_1 u' p'_i)^* = \sum_{i=1}^{N+1} u'_i u'^*.$$

This proves (2.2).

For the converse, we can use [HL00, Proposition 2.21]. We include the details. Consider the frame transforms Θ_i for the Parseval frame vectors ξ_i , Θ_i defined on \mathcal{H} for $i = 1, \dots, N$ and on $p'_r \mathcal{H}$ for $i = N+1$. We have, by Proposition 1.8, $\Theta_i^* \Theta_i = 1_{\mathcal{H}}$ for $i = 1, \dots, N$, $\Theta_{N+1}^* \Theta_{N+1} = 1_{p'_r \mathcal{H}}$. Note that $\Theta_{u'_i \xi_i} = \Theta_{\xi_i} u'^*$ for $i = 1, \dots, N+1$ and $\Theta_i^* \Theta_j = 0$ for $i \neq j$. So we have

$$\Theta_{\eta}^* \Theta_{\eta} = \left(\sum_{i=1}^{N+1} u'_i \Theta_i^* \right) \left(\sum_{i=1}^{N+1} \Theta_i u'^* \right) = \sum_{i,j=1}^{N+1} u'_i \Theta_i^* \Theta_j u'^* = \sum_{i=1}^{N+1} u'_i u'^* = I.$$

Hence η is a Parseval frame.

In the case when $q' \leq 1$, we have $q' \in \pi(G)'$ and, by Lemma 2.4, there exists a Parseval frame vector $\tilde{\eta}$ for \mathcal{H} such that $\eta = q' \tilde{\eta}$. Using the proof above we can find $\tilde{u}'_i \in \pi(G)'$ such that $\tilde{\eta} = \sum_{i=1}^{N+1} \tilde{u}'_i \xi_i$ and all the other properties. Then we can define $u'_i := q' \tilde{u}'_i$, $i = 1, \dots, N+1$, and a simple computation shows that the required properties are satisfied. The proof of uniqueness and the converse in this case is analogous to the one provided for the case $q' = 1$. \square

We can use the above parametrization result to characterize strongly disjoint (resp. unitary equivalent) Parseval frame vectors in terms of the given parametrization. Recall that two Parseval frame vectors η and ξ are unitary equivalent if and only if their analysis operators have the same range spaces, which in turn is equivalent to the condition that $\Theta_{\eta} \Theta_{\eta}^* = \Theta_{\xi} \Theta_{\xi}^*$.

Theorem 2.7. *Let G be a countable ICC group and let $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ be a unitary representation of the group G on the Hilbert space \mathcal{H} . Suppose ξ_i , $i = 1, \dots, N+1$ is a strongly*

disjoint $N + 1$ -tuple of Parseval frame vectors as in Theorem 2.2(i). Let $\eta, \zeta \in \mathcal{H}$, be two Parseval frame vectors for some subspaces of \mathcal{H} . Suppose

$$\eta = u'_1 \xi_1 + \cdots + u'_{N+1} \xi_{N+1}, \quad \zeta = v'_1 \xi_1 + \cdots + v'_{N+1} \xi_{N+1},$$

with $u'_i, v'_i \in \pi(G)'$, $i = 1, N + 1$, $u'_{N+1} p'_r = u'_{N+1}$, $v'_{N+1} p'_r = v'_{N+1}$. Then we have

(i) η and ζ are strongly disjoint if and only if

$$(2.3) \quad v'_1 u'^*_{N+1} + \cdots v'_{N+1} u'^*_{N+1} = 0.$$

(ii) η and ζ are unitary equivalent if and only if

$$(2.4) \quad [u'^*_1, \dots, u'^*_{N+1}]^t [u'_1, \dots, u'_{N+1}] = [v'^*_1, \dots, v'^*_{N+1}]^t [v'_1, \dots, v'_{N+1}],$$

where “ t ” represents the transpose of the row vector.

Proof. (i) Let $\psi, \psi' \in \mathcal{H}$. Then, by Proposition 1.4, η and ξ are disjoint if and only if

$$\begin{aligned} 0 &= \sum_{g \in G} \langle \pi(g) \eta, \psi \rangle \overline{\langle \pi(g) \zeta, \psi' \rangle} = \sum_{g \in G} \left\langle \pi(g) \left(\sum_{i=1}^{N+1} u'_i \xi_i \right), \psi \right\rangle \overline{\left\langle \pi(g) \left(\sum_{j=1}^{N+1} v'_j \xi_j \right), \psi' \right\rangle} = \\ &= \sum_{i,j=1}^{N+1} \sum_{g \in G} \langle \pi(g) \xi_i, u'^*_i \psi \rangle \overline{\langle \pi(g) \xi_j, v'^*_j \psi' \rangle} = (\text{since } \xi_i \text{ are mutually disjoint}) = \\ &= \sum_{i=1}^{N+1} \sum_{g \in G} \langle \pi(g) \xi_i, u'^*_i \psi \rangle \overline{\langle \pi(g) \xi_i, v'^*_i \psi' \rangle} = \\ &= (\text{since } \xi_i \text{ is a Parseval frame vector, } u'^*_{N+1} \psi, v'^*_{N+1} \psi \in p'_r \mathcal{H}) = \\ &= \sum_{i=1}^{N+1} \langle u'^*_i \psi, v'^*_i \psi \rangle = \left\langle \sum_{i=1}^{N+1} v'_i u'^*_i \psi, \psi' \right\rangle. \end{aligned}$$

Since $\psi, \psi' \in \mathcal{H}$ are arbitrary, the proof of (i) is complete.

(ii) Let Θ_i be the analysis operator for ξ_i . Then $\Theta_\eta = \sum_{i=1}^{N+1} \Theta_i u'^*_i$ and $\Theta_\zeta = \sum_{i=1}^{N+1} \Theta_i v'^*_i$. Then η and ζ are unitary equivalent if and only if $\Theta_\eta \Theta_\eta^* = \Theta_\zeta \Theta_\zeta^*$, i.e.

$$\sum_{i=1}^{N+1} \Theta_i u'^*_i \Theta_\eta^* = \sum_{i=1}^{N+1} \Theta_i v'^*_i \Theta_\zeta^*.$$

Since Θ_i have orthogonal range spaces, we have that the above equation holds if and only if

$$(2.5) \quad \Theta_i u'^*_i \Theta_\eta^* = \Theta_i v'^*_i \Theta_\zeta^*$$

for all $i = 1, \dots, N + 1$. Applying Θ_i^* to both sides of (2.5) and using that fact that $\Theta_i^* \Theta_i = I$ for $i = 1, \dots, N$, $\Theta_{N+1}^* \Theta_{N+1} = p'_r$, $u'_{N+1} p'_r = u'_{N+1}$ and $v'_{N+1} p'_r = v'_{N+1}$, we obtain that $u'^*_i \Theta_\eta^* = v'^*_i \Theta_\zeta^*$ for all $i = 1, \dots, N + 1$, i.e.,

$$\sum_{i=1}^N (u'^*_i u'_i - v'^*_i v'_i) \Theta_i^* = 0.$$

Apply the above left-side operator to $\Theta_j(\mathcal{H})$ and use the fact $\Theta_i^* \Theta_j = 0$ when $i \neq j$, we get $u'^*_i u'_i - v'^*_i v'_i = 0$ for all i, j . Hence we have

$$[u'^*_1, \dots, u'^*_{N+1}]^t [u'_1, \dots, u'_{N+1}] = [v'^*_1, \dots, v'^*_{N+1}]^t [v'_1, \dots, v'_{N+1}].$$

Conversely, if the above identity holds, then we clearly have

$$\sum_{i=1}^N (u_i'^* u_i' - v_i'^* v_i') \Theta_i^* = 0$$

for all i and so we have that (2.5) holds for all i . Therefore η and ζ are ununitary equivalent \square

We conclude this section by pointing out that we also have similar results as Theorem 2.6 and Theorem 2.7 for frame representations of arbitrary countable groups. The following lemma replaces Theorem 2.2 for the general group case. Note that, unlike the ICC group case, here we can not require that ξ_2, \dots, ξ_N are Parseval frame vectors for \mathcal{H} . Recall that the cyclic multiplicity for an subspace \mathcal{S} of operators on \mathcal{H} is the smallest cardinality k such that there exist vectors y_i ($i = 1, \dots, k$) with the property $\overline{\text{span}}\{Sy_i : S \in \mathcal{S}, i = 1, \dots, k\} = \mathcal{H}$.

Lemma 2.8. [HL08] *Let $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ be a unitary representation of a countable group G on the Hilbert space \mathcal{H} such that $\pi(G)'$ has cyclic multiplicity $N + 1$ (here N could be ∞). Assume ξ_1 is a Parseval frame vector for \mathcal{H} . Then there exist ξ_i for $i = 2, \dots, N + 1$ with the properties:*

- (i) $\{\pi(g)\xi_i : g \in G\}$ is a Parseval frame for $M_i := \overline{\text{span}}\{\pi(g)\xi_i : g \in G\}$;
- (ii) ξ_i ($i = 1, \dots, N + 1$) are mutually strongly disjoint;
- (iii) there is no non-zero Bessel vector which is strongly disjoint with all ξ_i .

One fact we used in the proof of Theorem 2.6 is that two ONB vectors for a unitary representation are linked by a unitary operator in the commutant of the representation, i.e. all the ONB vectors are unitarily equivalent. However, as we have already mentioned before this is no longer true in general for non-ONB vectors. The following characterizes the representations that have this property, and it is needed in order to prove our new parametrization result (Theorem 2.10) for general countable groups.

Lemma 2.9. [GH04] *Let $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ be a unitary representation of a countable group G on the Hilbert space \mathcal{H} and ξ be a Parseval frame vector for \mathcal{H} . Then the following statements are equivalent*

- (i) there is no non-zero Bessel vector which is strongly disjoint with all ξ ;
- (ii) $P_\xi = \Theta_\xi \Theta_\xi^* \in L(G)' \cap L(G)''$;
- (iii) a vector η is a Parseval vector for \mathcal{H} if and only if there exists a unitary $u' \in \pi(G)'$ such that $\eta = u'\xi$.

Theorem 2.10. *Let G be a countable group and let $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ be a unitary representation of the group G on the Hilbert space \mathcal{H} . Suppose ξ_i , $i = 1, \dots, N + 1$ are as in Lemma 2.8.*

(i) *Let $\eta \in \mathcal{H}$. Then η is a Parseval frame vector for \mathcal{H} if and only if there exist $u_i' \in \pi(G)'$, $i = 1, \dots, N + 1$, with $u_i' p_i' = u_i'$, $i = 1, \dots, N + 1$ (where p_i' is the projection onto the closed linear span of $\{\pi(g)\xi_i : g \in G\}$) such that*

$$\eta = u_1' \xi_1 + \dots + u_{N+1}' \xi_{N+1}$$

and $\sum_{i=1}^{N+1} u_i' u_i'^ = I$. Moreover these u_i' s are unique.*

(ii) *Let $\eta, \zeta \in \mathcal{H}$, be two Parseval frame vectors for \mathcal{H} such that*

$$\eta = u_1' \xi_1 + \dots + u_{N+1}' \xi_{N+1}, \quad \zeta = v_1' \xi_1 + \dots + v_{N+1}' \xi_{N+1},$$

with u_i and v_i satisfying the requirement as in (i). Then

(a) η and ζ are strongly disjoint if and only if

$$v'_1 u_1'^* + \dots v'_{N+1} u_{N+1}'^* = 0,$$

and

(b) η and ζ are unitary equivalent if and only if

$$[u_1'^*, \dots, u_{N+1}'^*]^t [u'_1, \dots, u'_{N+1}] = [v_1'^*, \dots, v_{N+1}'^*]^t [v'_1, \dots, v'_{N+1}].$$

Proof. We only give a sketch proof for the necessary part of (i). The rest is similar to the ICC group case.

Let η is a Parseval frame vector for \mathcal{H} . Then we have that $p'_\eta \sim p'_{\xi_1}$ in $L(G)'$ by Lemma 2.1. Since $L(G)'$ is a finite von Neumann algebra, we can find projections [KR97] $q'_i (i = 2, \dots, N+1)$ such that $p'_\eta, q'_i (i = 2, \dots, N+1)$ are mutually orthogonal and $q'_i \sim p'_{\xi_i}$ for $i = 2, \dots, N+1$. From Lemma 2.1, there exist Parseval frame vectors η_i for $M_i := \overline{\text{span}}\{\pi(g)\xi_i : g \in G\}$ such that $p'_{\eta_i} = q'_i (i = 2, \dots, N+1)$. Define unitary representation $\sigma : G \rightarrow \mathcal{U}(\mathcal{K})$ (where $\mathcal{K} = \mathcal{H} \oplus M_2 \oplus \dots \oplus M_{N+1}$) by

$$\sigma(g) = \pi(g) \oplus \pi(g)p_{\xi_2} \oplus \dots \oplus \pi(g)p_{\xi_{N+1}}$$

Then both $\tilde{\xi} := \xi_1 \oplus \dots \oplus \xi_{N+1}$ and $\tilde{\eta} := \eta_1 \oplus \dots \oplus \eta_{N+1}$ are Parseval frame vectors for \mathcal{K} . By Lemma 2.8 (iii) we have that there is no non-zero Bessel vector that is strongly disjoint with all ξ_i . This implies that there is no non-zero Bessel vector that is strongly disjoint with $\tilde{\xi}$. Thus, from Lemma 2.9, we get that there is a (unique) unitary operator $u' \in \sigma(G)'$ such that $\tilde{\eta} = u'\tilde{\xi}$. Let $u'_i = u'_{1,i}P'_i$ (where $[u'_{1,1}, \dots, u'_{1,N+1}]$ is the first row vector of u'). Then it can be checked that $\eta = u'_1\xi_1 + \dots + u'_{N+1}\xi_{N+1}$, and u'_i satisfy all the requirements listed in (i). \square

We make a final remark that all the results in this section remain valid when unitary representation is replaced by *projective unitary representations* of countable groups. In particular, Theorem 2.6 and Theorem 2.7 remain true for Gabor unitary representations. The interested reader can check (cf. [GH01, GH03, GH04, Grö01, Han04, Han08, HL08, Hei07]) for definitions and recent developments about projective unitary representations and Gabor representations.

3. PARSEVAL FRAMES AND SUBGROUPS

In this section we will only be interested in Parseval frame vectors ξ that have $\|\xi\|^2 = \frac{1}{N}$. We will assume in addition that there is a subgroup H of G of index N . The next proposition shows that in this situation we can find orthonormal bases for \mathcal{H} obtained by the action of the subgroup H .

Proposition 3.1. *Let G be a countable ICC group and let $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ be a unitary representation of the group G on the Hilbert space \mathcal{H} . Suppose $\xi \in \mathcal{H}$ is a Parseval frame vector for \mathcal{H} with $\|\xi\|^2 = \frac{1}{N}$ for some $N \in \mathbb{N}$. Assume in addition that there exists an ICC subgroup H of index $[G : H] = N$. Then there exists a Parseval frame vector η for \mathcal{H} with the property that $\sqrt{N}\{\pi(h)\eta \mid h \in H\}$ is an orthonormal basis for \mathcal{H} .*

Proof. By Theorem 1.9 we can assume that $\mathcal{H} = p'l^2(G)$ for some $p' \in L(G)'$ with $\text{tr}_{L(G)'}(p') = \frac{1}{N}$ and $\pi = \lambda$ restricted to $p'l^2(G)$. We claim that $\dim_{L(H)} \mathcal{H} = 1$.

Using [JS97, Proposition 2.3.5, Example 2.3.3, Proposition 2.2.1] we have

$$\dim_{L(H)} \mathcal{H} = \dim_{L(G)} \mathcal{H} \cdot [L(G) : L(H)] = N \dim_{L(G)} p'l^2(G) = N \text{tr}_{L(G)'}(p') \dim_{L(G)} l^2(G) = 1.$$

Thus (see [JS97, Chapter 2.2]) the Hilbert space \mathcal{H} considered as a module over $L(H)$, with the representation $\pi(h) = \lambda(h)p'$, $h \in H$, is isomorphic to the module $l^2(H)$, i.e., there exists an isometric isomorphism $\Phi : \mathcal{H} \rightarrow l^2(H)$ such that $\Phi\pi(h) = \lambda(h)\Phi$ for all $h \in H$.

Define $\eta := \frac{1}{\sqrt{N}}\Phi^{-1}(\delta_e)$. Then $\sqrt{N}\{\pi(h)\eta \mid h \in H\} = \Phi^{-1}\{\delta_h \mid h \in H\}$ so it is an orthonormal basis for \mathcal{H} .

We check that $\{\pi(g)\eta \mid g \in G\}$ is a Parseval frame for \mathcal{H} . Let $\{a_1, \dots, a_N\}$ be a complete set of representatives for the left cosets $\{gH \mid g \in G\}$. Let $v \in \mathcal{H}$. Then, since $\sqrt{N}\{\lambda(h)\eta \mid h \in H\}$ is an orthonormal basis, we have

$$\begin{aligned} \sum_{g \in G} |\langle v, \pi(g)\eta \rangle|^2 &= \sum_{i=1}^N \sum_{h \in H} |\langle v, \pi(a_i h)\eta \rangle|^2 = \sum_{i=1}^N \sum_{h \in H} |\langle \pi(a_i)^* v, \pi(h)\eta \rangle|^2 = \\ &= \sum_{i=1}^N \frac{1}{N} \|\pi(a_i)^* v\|^2 = 1. \end{aligned}$$

□

Remark 3.2. The condition $\|\xi\|^2 = \frac{1}{N}$ is essential. In other words, suppose η is a Parseval frame vector for the representation π of G on \mathcal{H} . Let H be a subgroup of G of index N , and suppose the family $\{\pi(h)\eta \mid h \in H\}$ is an orthogonal basis for the *whole* space \mathcal{H} . Then $\|\xi\|^2 = \frac{1}{N}$.

To see this, let $g_1 \dots g_N \in G$ be representatives of the left-cosets of H in G . We have on one hand, using the orthogonal basis, for all $x \in \mathcal{H}$:

$$\sum_{h \in H} |\langle \pi(h)\eta, x \rangle|^2 = \|\eta\|^2 \|x\|^2.$$

On the other hand, using the Parseval frame

$$\begin{aligned} \|x\|^2 &= \sum_{i=1}^N \sum_{h \in H} |\langle \pi(g_i)\pi(h)\eta, x \rangle|^2 = \sum_{i=1}^N |\langle \pi(h)\eta, \pi(g_i)^* x \rangle|^2 = \\ &= \sum_{i=1}^N \|\eta\|^2 \|\pi(g_i)^* x\|^2 = N \|x\|^2 \|\eta\|^2. \text{ Thus } \|\eta\|^2 = \frac{1}{N}. \end{aligned}$$

We saw in Theorem 2.2 that we can construct N strongly disjoint Parseval frame vectors for our Hilbert space \mathcal{H} . We want to see if we can do this in such a way that, by undersampling with the subgroup H , we have orthonormal bases (up to a multiplicative constant), as in Proposition 3.1. We prove that this is possible in the case when H is normal and has an element of infinite order.

Theorem 3.3. *Let G be a countable ICC group and let $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ be a unitary representation of the group G on the Hilbert space \mathcal{H} . Suppose there exists a Parseval frame vector $\xi \in \mathcal{H}$ with $\|\xi\|^2 = \frac{1}{N}$, $N \in \mathbb{Z}$. Assume in addition that H is a normal ICC subgroup of G with index $[G : H] = N$, and H contains elements of infinite order. Then there exist a strongly disjoint N -tuple η_1, \dots, η_N of Parseval frame vectors for \mathcal{H} such that for all $i = 1, N$, the family $\sqrt{N}\{\pi(h)\eta_i \mid h \in H\}$ is an orthonormal basis for \mathcal{H} .*

Proof. By Proposition 1.8, we can assume that π is the restriction of the left regular representation λ on $p'l^2(G)$, where p' is a projection in $L(G)'$, with $\text{tr}_{L(G)'}(p') = \frac{1}{N}$.

We will define some unitary operators u_i on $l^2(G)$ that will help us build the frame vectors η_i from just one such frame vector η given by Proposition 3.1.

Let $a_k, k = 0, N-1$ be a complete set of representatives for the cosets in G/H . Since H is normal, $a_k^{-1}H, k = 0, N-1$ forms a partition of G . We can take $a_0 = e$.

Define the functions $\varphi_j : G \rightarrow \mathbb{C}, \varphi_j(g) = e^{2\pi i \frac{kj}{N}}$ if $g \in a_k^{-1}H$. Note that

$$(3.1) \quad \sum_{k=0}^{N-1} \varphi_i(a_k^{-1}g) \overline{\varphi_j(a_k^{-1}g)} = 0, \quad (g \in G, i \neq j).$$

Indeed, if $g = a_r h$ for some $r \in \{0, \dots, N-1\}$ and $h \in H$, then $a_k^{-1}g, k = 0, N-1$ will lie in different sets of the partition $\{a_l^{-1}H\}_{l=0, N-1}$, because H is normal. Then

$$\sum_{k=0}^{N-1} \varphi_i(a_k^{-1}g) \overline{\varphi_j(a_k^{-1}g)} = \sum_{k=0}^{N-1} e^{2\pi i \frac{(i-j)k}{N}} = 0.$$

Also $\varphi_i(gh^{-1}) = \varphi_i(g)$, for all $g \in G, h \in H$, and, since H is normal $\varphi_i(hg) = \varphi_i(g)$, $i = 0, N-1, g \in G, h \in H$.

Define the operators u_j on $l^2(G)$ by $u_j \delta_g := \varphi_j(g) \delta_g$, for all $g \in G, j = 0, N-1$. Since u_j maps an ONB to an ONB, it is a unitary operator on $l^2(G)$.

Then for all $g \in G$, using (3.1),

$$\sum_{k=0}^{N-1} \lambda(a_k) u_i u_j^* \lambda(a_k)^* \delta_g = \sum_{k=0}^{N-1} \varphi_i(a_k^{-1}g) \overline{\varphi_j(a_k^{-1}g)} \delta_g = 0.$$

Also

$$u_i \lambda(h) \delta_g = \varphi_i(hg) \delta_{gh} = \varphi_i(g) \delta_{gh} = \lambda(h) u_i \delta_g,$$

so u_i commutes with $\lambda(h)$ for all $i = 0, N-1, h \in H$.

We want to compress the unitaries u_i to a subspace $p'_1 l^2(G)$ for some well chosen projection p'_1 in $L(G)'$. Take $h_0 \in H$, such that $h_0^n \neq e$ for all $n \in \mathbb{Z} \setminus \{0\}$. Then, with ρ the right-regular representation,

$$\langle \rho(h_0)^n \delta_e, \delta_e \rangle = \delta_n = \int_{\mathbb{T}} z^n d\mu(z), \quad (n \in \mathbb{Z}).$$

where μ is the Haar measure on \mathbb{T} . Then let p'_1 be the spectral projection $\chi_E(\rho(h_0))$, where E is a subset of \mathbb{T} of measure $\frac{1}{N}$. We have

$$\text{tr}_{L(G)'}(p'_1) = \langle p'_1 \delta_e, \delta_e \rangle = \langle \chi_E(\rho(h_0)) \delta_e, \delta_e \rangle = \int_{\mathbb{T}} \chi_E d\mu = \frac{1}{N}.$$

Also, for $i = 0, N-1, g \in G$,

$$u_i \rho(h_0) \delta_g = u_i \delta_{gh_0^{-1}} = \varphi_i(gh_0^{-1}) \delta_{gh_0^{-1}} = \varphi_i(g) \delta_{gh_0^{-1}} = \rho(h_0) u_i \delta_g,$$

so $u_i \rho(h_0) = \rho(h_0) u_i$, and therefore p'_1 commutes with all $u_i, i = 0, N-1$. In addition, since $p'_1 \in L(G)'$, it commutes with $\lambda(g)$ for all $g \in G$.

Then we compute for $i \neq j$

$$\sum_{k=0}^{N-1} (\lambda(a_k) p'_1) (p'_1 u_i p'_1) (p'_1 u_j p'_1) (\lambda(a_k) p'_1)^* \delta_g = p'_1 \sum_{k=0}^{N-1} \lambda(a_k) u_i u_j^* \lambda(a_k)^* \delta_g = 0.$$

The operators $\tilde{u}_i := p'_1 u_i p'_1$ are unitary on $p'_1 l^2(G)$ because p'_1 commutes with u_i . Also, $p'_1 u_i p'_1$ commute with $\lambda(h)$ on $p'_1 l^2(G)$.

We will couple these results with the following lemma to finish the proof.

Lemma 3.4. *Let G be an ICC group with an ICC subgroup H of index $[G : H] = N$. Let p'_1 be a projection in $L(G)'$ with $\text{tr}_{L(G)'}(p'_1) = \frac{1}{N}$ and let $\eta \in p'_1 l^2(G)$ such that $\sqrt{N}\{\lambda(h)\eta \mid h \in H\}$ is an orthonormal basis for $p'_1 l^2(G)$. Suppose $\tilde{u}_i, i = 0, N-1$ are unitary operators on $p'_1 l^2(G)$ such that \tilde{u}_i commutes with $\lambda(h)$ for all $h \in H$, and for some complete set of representatives a_0, \dots, a_{N-1} of the left-cosets in G/H ,*

$$\sum_{k=0}^{N-1} \lambda(a_k) \tilde{u}_i \tilde{u}_j^* \lambda(a_k)^* = 0, \quad (i \neq j).$$

Then the vectors $\tilde{u}_i \eta_0, i = 0, \dots, N-1$ have the following properties:

- (i) $\sqrt{N}\{\lambda(h)\tilde{u}_i \eta_0 \mid h \in H\}$ is an orthonormal basis for $p'_1 l^2(G)$ for all $i = 0, \dots, N-1$.
- (ii) $\tilde{u}_0 \eta_0, \dots, \tilde{u}_{N-1} \eta_0$ is a strongly disjoint N -tuple of Parseval frame vectors.

Proof. Since \tilde{u}_i commutes with $\lambda(h)$ for all $h \in H$, property (i) follows immediately from the hypothesis. This implies also that $\tilde{u}_i \eta$ is a Parseval frame vector for $p'_1 l^2(G)$ (see the proof of Proposition 3.1).

To check the strong disjointness, let Θ_i be the frame transform of the vector $\tilde{u}_i \eta_0$. Let $\Theta_0^H : p'_1 l^2(G) \rightarrow l^2(H)$ be the frame transform for the $\frac{1}{\sqrt{N}}$ -orthonormal basis $\{\lambda(h)\eta \mid h \in H\}$. Then $\Theta_0^{H*} \Theta_0^H = \frac{1}{N} I$. We have for $v \in p'_1 l^2(G)$:

$$\begin{aligned} \Theta_i(v) &= (\langle v, \lambda(g)\tilde{u}_i \eta \rangle)_{g \in G} = ((\langle v, \lambda(a_k)\lambda(h)\tilde{u}_i \eta \rangle)_{h \in H})_{k=0, N-1} = \\ &= ((\langle \tilde{u}_i^* \lambda(a_k)^* v, \lambda(h)\eta_0 \rangle)_{h \in H})_{k=0, N-1}. \end{aligned}$$

Then for $v, v' \in p'_1 l^2(G)$,

$$\begin{aligned} \langle \Theta_i(v), \Theta_j(v') \rangle &= \sum_{k=0}^{N-1} \sum_{h \in H} \langle \tilde{u}_i^* \lambda(a_k)^* v, \lambda(h)\eta \rangle \overline{\langle \tilde{u}_j^* \lambda(a_k)^* v', \lambda(h)\eta \rangle} = \\ &= \sum_{k=0}^{N-1} \langle \Theta_0^H(\tilde{u}_i^* \lambda(a_k)^* v), \Theta_0^H(\tilde{u}_j^* \lambda(a_k)^* v') \rangle = \sum_{k=0}^{N-1} \langle \lambda(a_k) \tilde{u}_j \Theta_0^{H*} \Theta_0^H \tilde{u}_i^* \lambda(a_k)^* v, v' \rangle = 0. \end{aligned}$$

This proves that the frames are strongly disjoint. \square

Returning to the proof of the theorem, we see that we can apply Lemma 3.4. Let η be a Parseval frame vector in $p'_1 l^2(G)$ such that $\sqrt{N}\{\lambda(h)\eta \mid h \in H\}$ is an ONB for $p'_1 l^2(G)$. It can be obtained from Proposition 3.1. Then, using Lemma 3.4, we get that $\eta_i := p'_1 u_i p'_1 \eta$ form a strongly disjoint N -tuple of Parseval frames and $\sqrt{N}\{\lambda(h)\eta_i \mid h \in H\}$ are ONBs for $p'_1 l^2(G)$. Then we can move everything onto our space \mathcal{H} , because $\text{tr}_{L(G)'}(p'_1) = \text{tr}_{L(G)'}(p')$ so the projections p'_1 and p' are equivalent in $L(G)'$ and the representations λ on $p'_1 l^2(G)$ and π on \mathcal{H} are equivalent. \square

Remark 3.5. (i) There exist ICC groups with ICC subgroups of any finite index. For example, let F_N the free group on N generators and $p \in \mathbb{N}$. Also, let $\phi : F_N \rightarrow \mathbb{Z}_p$ a surjective group morphism. Then $H := \text{Ker} \phi$ is a (free, thus ICC) normal subgroup of F_N of finite index.

(ii) There exist ICC groups without finite index proper subgroups. For example, let F

be the Thompson's group and F' its commutator. Both groups are ICC (see e.g. [Jol98]). Moreover, F' is infinite and simple. By a classic group theoretical argument an infinite simple group cannot have finite index proper subgroups. Indeed, let G be infinite simple and H of finite index k in G . Then there exists a group morphism from G to the (finite) group of permutations of the set of right cosets $X := \{Hg_i | i = 1, \dots, k\}$. This is given by $g \rightarrow \alpha_g$ where $\alpha_g(Hg_i) = Hg_i g$. Because G is infinite the kernel of the above morphism must be non-trivial. Moreover, because G is simple the kernel must be all of G , i.e. $\alpha_{g_i}(H) = H = Hg_i$ for all $i = 1, \dots, k$. Hence H is not proper.

(iii) If G is ICC and H is a finite index subgroup of G then H is ICC. Indeed, let $G = \cup_{j=1}^k c_j H$, where c_j are distinct left cosets representatives. If for some $h \in H$ the conjugacy class $\{ghg^{-1} | g \in H\}$ is finite then the set $\{c_j gh(c_j g)^{-1} | g \in H, j = 1, \dots, k\}$ is finite. Notice $\{chc^{-1} | c \in G\} \subset \{c_j gh(c_j g)^{-1} | g \in H, j = 1, \dots, k\}$. However, the conjugacy class of h in G is infinite as G is ICC.

(iv) There exist ICC groups with all elements of finite order, e.g. the Burnside groups of large enough exponents (see [Ol'80]).

Proposition 3.6. *Let G be a countable ICC group and let $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ be a unitary representation of the group G on the Hilbert space \mathcal{H} . Suppose $\xi \in \mathcal{H}$ is a Parseval frame vector for \mathcal{H} with $\|\xi\|^2 = \frac{M}{N}$ for some $M, N \in \mathbb{N}$. Assume in addition that there exists a normal ICC subgroup H of index $[G : H] = N$ such that H contains elements of infinite order. Then there exists $K := \lfloor \frac{N}{M} \rfloor$ strongly disjoint Parseval frame vectors η_i for \mathcal{H} , $i = 1, \dots, K$ such that $\{\pi(h)\eta_i | h \in H\}$ is an orthogonal family (in a subspace of \mathcal{H}) for all $i = 1, \dots, K$.*

Proof. Consider a projection p' in $L(G)'$ of trace $\text{tr}_{L(G)'}(p') = \frac{1}{N}$. Using Theorem 3.3 we can find ξ_1, \dots, ξ_N strongly disjoint Parseval frame vectors for $\mathcal{H}_{1/N} := p'l^2(G)$ with the representation $\pi_{1/N} := p'\lambda$, such that for all $i = 1, \dots, N$, $\{\pi(h)\xi_i | h \in H\}$ is an orthogonal basis for $\mathcal{H}_{1/N}$.

Then consider the representation $\pi_{1/N}^M$ on $\mathcal{H}_{1/N}^M$ with the strongly disjoint Parseval frame vectors $\eta_i := \xi_{(M-1)i+1} \oplus \dots \oplus \xi_{Mi}$, $i = 1, \dots, K$.

Using Proposition 1.8 we have that $\pi_{1/N}^M$ is equivalent to a subrepresentation of the left-regular representation, corresponding to a projection of trace $\|\xi_1 \oplus \dots \oplus \xi_M\|^2 = \frac{M}{N}$. But the same is true for the representation π on \mathcal{H} . Therefore the two representations are equivalent and the vectors η_i can be mapped into \mathcal{H} to obtain the conclusion. \square

Remark 3.7. Suppose the hypotheses of Theorem 3.3 are satisfied, with $N \geq 2$. Then we can construct uncountably many inequivalent Parseval frame vectors η for \mathcal{H} with the property that $\{\pi h(\eta) | h \in H\}$ is a Riesz basis for \mathcal{H} .

To see this, use Theorem 3.3 to obtain strongly disjoint Parseval frame vectors η_1, \dots, η_N for \mathcal{H} , such that $\{\pi(h)\eta_i | h \in H\}$ is an orthogonal basis for \mathcal{H} , for all $i = 1, N$.

Then take $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$, and $|\alpha| \neq |\beta|$. Using Theorem 2.6, we obtain that $\eta_{\alpha,\beta} := \alpha\eta_1 + \beta\eta_2$ is a Parseval frame vector for \mathcal{H} .

Since η_1 and η_2 generate orthogonal bases under the action of H , there is a unitary $u \in \pi(H)'$ such that $u\eta_1 = \eta_2$. Then $\eta_{\alpha,\beta} = (\alpha + \beta u)\eta_1$. Since u is unitary and $|\alpha| \neq |\beta|$ it follows that $\alpha + \beta u$ is invertible. Therefore $\{\pi(h)\eta_{\alpha,\beta} | h \in H\} = \{(\alpha + \beta u)\pi(h)\eta_1 | h \in H\}$ is a Riesz basis for \mathcal{H} .

It remains to see when two such vectors $\eta_{\alpha,\beta}, \eta_{\alpha',\beta'}$ are equivalent. Using Theorem 2.7 we see that this happens only if $|\alpha| = |\alpha'|$, $|\beta| = |\beta'|$ and $\overline{\alpha}\beta = \overline{\alpha'}\beta'$, i.e., $(\alpha, \beta) = c(\alpha', \beta')$ for some $c \in \mathbb{C}$ with $|c| = 1$. Since we can find uncountably many pairs (α, β) such that no two such pairs satisfy this condition, it follows that we can construct uncountably many inequivalent Parseval frame vectors $\eta_{\alpha,\beta}$ that satisfy the given conditions.

Finally we discuss how our results fit in the recent effort on the Feichtinger's frame decomposition conjecture. It was recently discovered (in particular, by Pete Casazza and his collaborators) that the famous intractable 1959 Kadison-Singer Problem in C^* -algebras is equivalent to fundamental open problems in a dozen different areas of research in mathematics and engineering (cf. [CFTW06, CT06]). Particularly, the KS-problem is equivalent to the Feichtinger's problem which asks whether every bounded frame (i.e. the norms of the vectors in the frame sequence are bounded from below) can be written as a finite union of Riesz sequences. Since this question is intractable in general, much of the effort has been focused on special classes of frames. One natural and interesting class to consider is the class of frames obtained by group representations [See open problems posted at the 2006 "The Kadison-Singer Problem" workshop]. Unfortunately, except for a very few cases (e.g., Gabor frames associated with rational lattices [CCLV05]) very little is known so far even for this special class. Particularly, it is unknown whether for every (frame) unitary representation we can always find one frame vector which is "Riesz sequence" decomposable. Therefore the results obtained in this section certainly addressed some aspects of the research effort in this direction. In particular, we have the following as a consequence of our main result.

Proposition 3.8. *Let G be a countable ICC group and assume that there exists an ICC subgroup H of index $[G : H] = N$. Then*

(i) *If $p' \in L(G)'$ is any projection such that $\text{tr}_{L(G)'}(p') \geq \frac{1}{N}$, then there exist a Parseval frame vector η for the subrepresentation $\pi := L|_{p'}$ such that $\{\pi(g)\eta : g \in G\}$ is a finite union of Riesz sequences.*

(ii) *For any ONB vector for the left regular representation L and any α such that $1 > \alpha \geq \frac{1}{N}$, there exists a projection $p' \in L(G)'$ such that $\text{tr}_{L(G)'}(p') = \alpha$ and $\{p'L(g)\psi : g \in G\}$ is a finite union of Riesz sequences.*

Proof. (i) Since $L(G)'$ is a factor von Neumann algebra, there exists a subprojection q' of p' such that $\text{tr}_{L(G)'}(q') = \frac{1}{N}$. By Proposition 3.1, there exists a Parseval frame vector, say η_1 , for the representation $\pi|_{q'}$ such that $\{\sqrt{N}\pi(h)\eta_1 : h \in H\}$ is orthonormal. By Lemma 2.4, we can "dilate" η_1 to a Parseval frame vector η for π . Let $\eta_2 = (p' - q')\eta$. Then for any sequence $\{c_h\}_{h \in H}$ (finitely many of them are non-zero) we have

$$\left\| \sum_{h \in H} c_h \pi(h)\eta \right\|^2 = \left\| \sum_{h \in H} c_h \pi(h)\eta_1 \right\|^2 + \left\| \sum_{h \in H} c_h \pi(h)\eta_2 \right\|^2 \geq \left\| \sum_{h \in H} c_h \pi(h)\eta_1 \right\|^2$$

since $\pi(G)\eta_1$ and $\pi(G)\eta_2$ are orthogonal. Thus $\{\pi(h)\eta : h \in H\}$ is a Riesz sequence as $\{\pi(h)\eta_1 : h \in H\}$ is Riesz.

(ii) Let $r' \in L(G)'$ be a projection such that $\text{tr}_{L(G)'}(r') = \alpha$. Then by part (i) there exists a Parseval frame vector ξ such that $\{\sigma(h)\xi : h \in H\}$ is a Riesz sequence, where $\sigma = L|_{r'}$. We will show that there exists a projection $p' \in L(G)'$ such that $\{p'L(g)\psi : g \in G\}$ and $\{\sigma(g)\xi : g \in G\}$ are unitarily equivalent, and this will imply that $\{p'L(h)\psi : h \in H\}$ is a Riesz sequence.

In fact, again by Lemma 2.4, there exists ONB vector $\tilde{\psi}$ for the left regular representation L such that $r'\tilde{\psi} = \xi$. Since both ψ and $\tilde{\psi}$ are ONB vectors for L we have that there exists a unitary operator $u' \in L(G)'$ such that $\psi = u'\tilde{\psi}$. Let $p' = u'r'u'^*$. Then $p' \in L(G)'$ is a projection such that $\text{tr}_{L(G)'}(p') = \text{tr}_{L(G)'}(r') = \alpha$. Moreover,

$$p'L(g)\psi = L(g)p'\psi = L(g)(u'r'u'^*)u'\tilde{\psi} = u'L(g)r'\tilde{\psi} = u'L(g)\xi = u'\sigma(g)\xi$$

for all $g \in H$. Hence $\{p'L(g)\psi : g \in G\}$ and $\{\sigma(g)\xi : g \in G\}$ are unitarily equivalent, as claimed and so we completed the proof. \square

There are some interesting special cases. For example, as we mentioned in Remark 3.5 if G is a free group with more than one generator, then we can find $N_k \rightarrow \infty$ such that there exist ICC subgroups H_k having the property $[G : H_k] = N_k$. Thus we have the following corollary which for the free group case answered affirmatively one of two open problems posted by Deguang Han at the 2006 “The Kadison-Singer Problem” workshop.

Corollary 3.9. *Let G be a free group with more than one generator. Then*

(i) *For any non-zero projection $p' \in L(G)'$, there exists a Parseval frame vector η for the subrepresentation $\pi := L|_{p'}$ such that $\{\pi(g)\eta : g \in G\}$ is a finite union of Riesz sequences.*

(ii) *For any ONB vector for the left regular representation L , and any $\alpha > 0$, there exists a projection $p' \in L(G)'$ such that $\text{tr}_{L(G)'}(p') = \alpha$ and $\{p'L(g)\psi : g \in G\}$ is a finite union of Riesz sequences. This sequence will be orthogonal when $\alpha = \frac{1}{N}$ for some $N \in \mathbb{N}$.*

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